Asian-European Journal of Mathematics
Vol. 19, No. 9 (2026) 2550144 (10 pages)

DOI: 10.1142/S179355712550144X

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Counting conjugacy classes and automorphism-conjugacy classes of ZM-groups

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Communicated by A. R. Rajan
Received March 3, 2025
Revised September 21, 2025
Accepted November 14, 2025
Published December 2, 2025

In this paper, we count the number of conjugacy and automorphism-conjugacy classes of elements of a ZM-group. The size of a conjugacy class with respect to these two equivalence relations is also counted.

Keywords: Conjugacy class; automorphism-conjugacy class; ZM-group; group action; Burnside's lemma; Menon's identity.

AMS Subject Classification: 20D60, 20D45, 11A25

1. Introduction

Given a finite group G, we denote by $\mathrm{Inn}(G)$ and $\mathrm{Aut}(G)$ the inner automorphism group and the automorphism group of G. Then the orbits of G under the natural actions of $\mathrm{Inn}(G)$ and $\mathrm{Aut}(G)$ are called the *conjugacy classes* and the *automorphism-conjugacy classes* of G. They are very useful in the study of groups, especially in representation theory. The numbers of these orbits are usually denoted by $k(G)$ and $k'(G)$, respectively, and represent two important invariants of G. Note that $k(G)$ is equal to the number of complex irreducible characters of G.

In what follows, we will determine these numbers and the sizes of conjugacy classes for ZM-groups. Recall that a *ZM-group* is a finite group with all Sylow subgroups cyclic. By [5], such a group is of type

$$\mathrm{ZM}(m,n,r) = \langle a, b \mid a^m = b^n = 1,\ b^{-1}ab = a^r \rangle,$$

where the triple (m,n,r) satisfies the conditions

$$(m,n) = (m, r-1) = 1 \quad \text{and} \quad r^n \equiv 1 \pmod m.$$

It is clear that $|\mathrm{ZM}(m,n,r)| = mn$ and $Z(\mathrm{ZM}(m,n,r)) = \langle b^d \rangle$, where d is the multiplicative order of r modulo m, i.e.

$$d = o_m(r) = \min\{k \in \mathbb{N}^* \mid r^k \equiv 1 \pmod m\}.$$

The subgroups of $\mathrm{ZM}(m,n,r)$ have been completely described in [2]. Set

$$L = \left\{ (m_1, n_1, s) \in \mathbb{N}^3 \mid m_1|m, n_1|n, s < m_1, m_1|s\frac{r^n - 1}{r^{n_1} - 1} \right\}.$$

Then there is a bijection between L and the subgroup lattice $L(\mathrm{ZM}(m,n,r))$ of $\mathrm{ZM}(m,n,r)$, namely, the function that maps a triple $(m_1,n_1,s) \in L$ to the subgroup $H_{(m_1,n_1,s)}$ defined by

$$H_{(m_1,n_1,s)} = \bigcup_{k=1}^{\frac{n}{n_1}} (b^{n_1}a^s)^k \langle a^{m_1} \rangle = \langle a^{m_1}, b^{n_1}a^s \rangle.$$

Note that

— $|H_{(m_1,n_1,s)}| = \frac{mn}{m_1 n_1}$, for any s satisfying $(m_1,n_1,s) \in L$;
— $H_{(m_1,n_1,s)}$ is normal in $\mathrm{ZM}(m,n,r)$ if and only if $m_1 \mid r^{n_1} - 1$ and $s = 0$ (see [10]);
— $H_{(m_1,n_1,s)}$ is cyclic if and only if $\frac{m}{m_1} | r^{n_1} - 1$ (see [11]);
— two subgroups of $\mathrm{ZM}(m,n,r)$ are conjugate if and only if they have the same order (see [1]).

Our approach uses some basic tools of group action theory, such as the orbit-stabilizer theorem and the Burnside's lemma, and involves modular arithmetic.

Orbit-Stabilizer theorem. *Let G be a finite group acting on a finite set X and $x \in X$. Denote by*

$$O_x = \{g \circ x \mid g \in G\} \quad \text{and} \quad \mathrm{Stab}_G(x) = \{g \in G \mid g \circ x = x\},$$

the orbit and the stabilizer of x in G. Then

$$|O_x| = [G : \mathrm{Stab}_G(x)] = \frac{|G|}{|\mathrm{Stab}_G(x)|}.$$

Burnside's lemma. *Let G be a finite group acting on a finite set X and*

$$\mathrm{Fix}(g) = \{x \in X \mid g \circ x = x\}, \quad \text{for } g \in G.$$

Then the number of distinct orbits is

$$\frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)|.$$

Most of our notations are standard and will not be repeated here. Basic definitions and results on group theory can be found in [5, 9]. For number theoretic notions, we refer the reader to [6, 8].

2. Main Results

Throughout this section, we will use G to denote $\mathrm{ZM}(m,n,r)$. The general form of automorphisms of a metacyclic group had been determined in the main result of [3]. For ZM-groups, in [4], we obtained the following theorem.

Theorem 2.1. *Each automorphism of* $\mathrm{ZM}(m,n,r)$ *is given by*

$$b^u a^v \mapsto b^{yu} a^{x_1 v + x_2 [u]_r}, \quad u,v \geq 0,$$

for a unique triple of integers (x_1,x_2,y) *such that*

$$0 \leq x_1, x_2 < m, \quad (x_1,m)=1, \quad 0 \leq y < n \quad \text{and} \quad y \equiv 1 \pmod{d}, \tag{1}$$

where

$$[u]_r = \begin{cases} 1 + r + \cdots + r^{u-1}, & u > 0, \\ 0, & u = 0. \end{cases}$$

In particular, we have

$$|\mathrm{Aut}(\mathrm{ZM}(m,n,r))| = m\varphi(m)\frac{n}{d}.$$

We are now able to compute the number of automorphism-conjugacy classes of G.

Theorem 2.2. *The following equality holds*:

$$k'(\mathrm{ZM}(m,n,r)) = \frac{d}{m\varphi(m)} \sum_{(x_1,x_2,y)\in A} \frac{(m, x_1 - 1)}{\left[\frac{n}{(n,y-1)}, o_{\frac{(m,x_1-1)}{(m,x_1-1,x_2)}}(r)\right]}, \tag{2}$$

where A is the set of triples (x_1,x_2,y) satisfying the conditions (1).

Proof. We will apply Burnside's lemma to the natural action of

$$\mathrm{Aut}(G) = \{\varphi_{(x_1,x_2,y)} \,|\, (x_1,x_2,y) \in A\},$$

on G. Let $(x_1,x_2,y) \in A$ and

$$\mathrm{Fix}(\varphi_{(x_1,x_2,y)}) = H_{(m_1,n_1,s)} = \langle a^{m_1}, b^{n_1}a^s \rangle,$$

where $(m_1,n_1,s) \in L$. Since $H_{(m_1,n_1,s)}$ is the join of its subgroups $\langle a^{m_1} \rangle$ and $\langle b^{n_1}a^s \rangle$, $m_1 | m$ and $n_1 | n$ are minimal with the property that there is $0 \leq s < m_1$ such that $m_1 | s\frac{[n]_r}{[n_1]_r}$ and both a^{m_1} and $b^{n_1}a^s$ are fixed points of $\varphi_{(x_1,x_2,y)}$. We have

$$a^{m_1} \in \mathrm{Fix}(\varphi_{(x_1,x_2,y)}) \Leftrightarrow a^{x_1 m_1} = a^{m_1} \Leftrightarrow m \,|\, m_1(x_1 - 1) \Leftrightarrow \left.\frac{m}{(m,x_1-1)}\right| m_1,$$

which leads to $m_1 = \frac{m}{(m,x_1-1)}$ and

$$b^{n_1}a^s \in \mathrm{Fix}(\varphi_{(x_1,x_2,y)}) \Leftrightarrow b^{yn_1}a^{x_1s+x_2[n_1]_r} = b^{n_1}a^s \Leftrightarrow \begin{cases} n \mid n_1(y-1), \\ m \mid s(x_1-1)+x_2[n_1]_r. \end{cases}$$

Thus n_1 is minimal such that $\frac{n}{(n,y-1)}|n_1$ and there is $0 \le s < m_1$ with

$$\begin{cases} m \mid s(x_1-1)\dfrac{[n]_r}{[n_1]_r}, \\ m \mid s(x_1-1)+x_2[n_1]_r. \end{cases}$$

Since

$$m \mid s(x_1-1)+x_2[n_1]_r \;\Rightarrow\; m \mid s(x_1-1)\frac{[n]_r}{[n_1]_r},$$

it follows that n_1 is minimal such that $\frac{n}{(n,y-1)}|n_1$ and

$$m \mid s(x_1-1)+x_2[n_1]_r,$$

for some $0 \le s < m_1$, i.e. the congruence

$$(x_1-1)s \equiv -x_2[n_1]_r \pmod{m},$$

has solutions. This is equivalent to

$$(m, x_1-1) \mid x_2[n_1]_r,$$

i.e.

$$\left.\frac{(m,x_1-1)}{(m,x_1-1,x_2)}\right| r^{n_1}-1,$$

which means that

$$o_{\frac{(m,x_1-1)}{(m,x_1-1,x_2)}}(r) \mid n_1.$$

Thus

$$n_1 = \left[\frac{n}{(n,y-1)},\; o_{\frac{(m,x_1-1)}{(m,x_1-1,x_2)}}(r)\right]$$

and

$$|\mathrm{Fix}(\varphi_{(x_1,x_2,y)})| = \frac{mn}{m_1n_1} = \frac{n(m,x_1-1)}{\left[\frac{n}{(n,y-1)},\; o_{\frac{(m,x_1-1)}{(m,x_1-1,x_2)}}(r)\right]}.$$

Then we get

$$\begin{aligned} k'(G) &= \frac{d}{mn\varphi(m)} \sum_{(x_1,x_2,y)\in A} \frac{n(m,x_1-1)}{\left[\frac{n}{(n,y-1)},\; o_{\frac{(m,x_1-1)}{(m,x_1-1,x_2)}}(r)\right]} \\ &= \frac{d}{m\varphi(m)} \sum_{(x_1,x_2,y)\in A} \frac{(m,x_1-1)}{\left[\frac{n}{(n,y-1)},\; o_{\frac{(m,x_1-1)}{(m,x_1-1,x_2)}}(r)\right]}, \end{aligned}$$

as desired. $\square$

In the notation in the proof of Theorem 2.2, we observe that

$$\frac{n}{(n, y-1)} \le n_1 \le n,$$

which leads to

$$(m, x_1 - 1) \le |\mathrm{Fix}(\varphi_{(x_1,x_2,y)})| \le (m, x_1 - 1)(n, y-1).$$

By summing over all $(x_1, x_2, y) \in A$ and using the well-known Menon's identity [7], we get

$$\tau(m) \le k'(G) \le \frac{d^2}{n} f\left(\frac{n}{d}\right) \tau(m),$$

where f is the arithmetic function defined by

$$f(\alpha) = \sum_{\beta=0}^{\alpha-1} (\alpha, \beta).$$

Note that if $\alpha = p_1^{\varepsilon_1}, \ldots, p_k^{\varepsilon_k}$ is the decomposition of α as a product of prime factors, then

$$f(\alpha) = \alpha \prod_{i=1}^{k} \left(\varepsilon_i + 1 - \frac{\varepsilon_i}{p_i}\right).$$

A simple formula for $k'(G)$ is obtained if n is a prime number.

Corollary 2.3. *If n is a prime number, then the following equality holds*:

$$k'(\mathrm{ZM}(m, n, r)) = n - 1 + \tau(m). \tag{3}$$

Proof. Since n is prime, we obtain $d = n$ and $y = 1$. By (2), it follows that

$$k'(G) = \frac{d}{m\varphi(m)} \sum_{\substack{0 \le x_1, x_2 < m \\ (x_1, m) = 1}} \frac{(m, x_1 - 1)}{o_{\frac{(m, x_1-1)}{(m, x_1-1, x_2)}}(r)}.$$

Moreover, we have

$$n_1 = o_{\frac{(m,x_1-1)}{(m,x_1-1,x_2)}}(r) = \begin{cases} 1 & \text{if } (m, x_1 - 1) \mid x_2, \\ n & \text{if } (m, x_1 - 1) \nmid x_2 \end{cases}$$

and so

$$|\mathrm{Fix}(\varphi_{(x_1,x_2,y)})| = \begin{cases} n(m, x_1 - 1) & \text{if } (m, x_1 - 1) \mid x_2, \\ (m, x_1 - 1) & \text{if } (m, x_1 - 1) \nmid x_2. \end{cases}$$

Then

$$
\begin{aligned}
k'(G) &= \frac{1}{m\varphi(m)} \sum_{\substack{0\le x_1<m\\(x_1,m)=1}} \left(\sum_{0\le x_2<m} |\mathrm{Fix}(\varphi_{(x_1,x_2,y)})| \right)\\
&= \frac{1}{m\varphi(m)} \sum_{\substack{0\le x_1<m\\(x_1,m)=1}} \left(\sum_{\substack{0\le x_2<m\\(m,x_1-1)\,\mid\, x_2}} n(m,x_1-1) + \sum_{\substack{0\le x_2<m\\(m,x_1-1)\,\nmid\, x_2}} (m,x_1-1) \right)\\
&= \frac{1}{m\varphi(m)} \sum_{\substack{0\le x_1<m\\(x_1,m)=1}} \left[mn + (m,x_1-1)\left(m - \frac{m}{(m,x_1-1)}\right)\right]\\
&= \frac{1}{m\varphi(m)} \left(mn\varphi(m) + m \sum_{\substack{0\le x_1<m\\(x_1,m)=1}} (m,x_1-1) - m\varphi(m) \right)\\
&= \frac{1}{m\varphi(m)} (mn\varphi(m) + m\varphi(m)\tau(m) - m\varphi(m))\\
&= n - 1 + \tau(m),
\end{aligned}
$$

completing the proof. □

In particular, for $n = 2$ and $r = m-1$, we have $\mathrm{ZM}(m,n,r) \cong D_{2m}$, the dihedral group of order $2m$. Thus, the equality (3) becomes

$$k'(D_{2m}) = \tau(m) + 1, \quad \text{for all odd integers } m \ge 3. \tag{4}$$

An example of applying the formula (2) is presented in the following example.

Example 2.4. For the dicyclic group $\mathrm{Dic}_3 = \mathrm{ZM}(3,4,2) = \mathrm{SmallGroup}(12,1)$, we have $d = 2$ and (2) leads to

$$k'(\mathrm{Dic}_3) = 5.$$

Next, we will focus on computing the number of conjugacy classes of G.

Theorem 2.5. *The following equality holds*:

$$k(\mathrm{ZM}(m,n,r)) = \frac{n}{md} \sum_{\alpha=0}^{d-1} \sum_{\beta=0}^{m-1} \frac{(m, r^\alpha - 1)}{o_{\frac{(m,r^\alpha-1)}{(m,r^\alpha-1,\beta)}}(r)}. \tag{5}$$

Proof. We have

$$\mathrm{Inn}(G) = \{\varphi_{b^\alpha a^\beta} \mid \alpha = \overline{0,d-1},\ \beta = \overline{0,m-1}\},$$

where $\varphi_{b^\alpha a^\beta}$ is the conjugation by $b^\alpha a^\beta$. In the notation in Theorem 2.2, it is easy to see that

$$\varphi_{b^\alpha a^\beta} = \varphi_{(r^\alpha, \beta(1-r), 1)}$$

and so

$$|\mathrm{Fix}(\varphi_{b^\alpha a^\beta})| = \frac{n(m, r^\alpha - 1)}{o_{\frac{(m,r^\alpha-1)}{(m,r^\alpha-1,\beta)}}(r)}.$$

Then (5) follows by applying Burnside's lemma to the natural action of $\mathrm{Inn}(G)$ on G. □

We remark that

$$o_{\frac{(m,r^\alpha-1)}{(m,r^\alpha-1,\beta)}}(r) = 1 \Leftrightarrow \frac{(m, r^\alpha - 1)}{(m, r^\alpha - 1, \beta)} = 1 \Leftrightarrow (m, r^\alpha - 1) \mid \beta. \tag{6}$$

Also, for $(m, r^\alpha - 1) \nmid \beta$ we have

$$\frac{1}{d} \leq \frac{1}{o_{\frac{(m,r^\alpha-1)}{(m,r^\alpha-1,\beta)}}(r)} \leq \frac{1}{p}, \tag{7}$$

where p is the smallest prime divisor of d. Then, writing

$$k(G) = \frac{n}{md} \sum_{\alpha=0}^{d-1} (m, r^\alpha - 1) \left(\sum_{\substack{0 \leq \beta < m-1 \\ (m,r^\alpha-1) \mid \beta}} \frac{1}{o_{\frac{(m,r^\alpha-1)}{(m,r^\alpha-1,\beta)}}(r)} + \sum_{\substack{0 \leq \beta < m-1 \\ (m,r^\alpha-1) \nmid \beta}} \frac{1}{o_{\frac{(m,r^\alpha-1)}{(m,r^\alpha-1,\beta)}}(r)} \right)$$

and using (6) and (7), we get

$$\frac{n}{d}\left(d - 1 + \frac{S}{d}\right) \leq k(G) \leq \frac{n}{d}\left(d - \frac{d}{p} + \frac{S}{p}\right), \tag{8}$$

where

$$S = \sum_{\alpha=0}^{d-1} (m, r^\alpha - 1).$$

Note that if d is prime, then $p = d$ and (8) leads to the equality

$$k(G) = \frac{n}{d}\left(d - 1 + \frac{S}{d}\right). \tag{9}$$

By considering the natural action of $\mathbb{Z}_m^\times$ on $\mathbb{Z}_m$, we have

$$S = \sum_{\alpha=0}^{d-1} |\mathrm{Fix}(\hat{r}^\alpha)|,$$

implying that

$$m + d - 1 \leq S \leq \sum_{\hat{a} \in \mathbb{Z}_m^\times} |\mathrm{Fix}(\hat{a})| = \varphi(m)\tau(m).$$

These inequalities, together with (8), show that

$$\frac{n(m+d^2-1)}{d^2} \le k(G) \le \frac{n[\varphi(m)\tau(m)+d(p-1)]}{dp}.$$

Again, if n is prime, then $d=n$ is also prime and (9) gives a simple formula for $k(G)$.

Corollary 2.6. *If n is a prime number, then the following equality holds*:

$$k(\mathrm{ZM}(m,n,r)) = n-1+\frac{1}{n}\sum_{\alpha=0}^{n-1}(m,r^\alpha-1).$$

In particular, for $n=2$ and $r=m-1$, we get

$$k(D_{2m}) = \frac{m+3}{2}, \quad \text{for all odd integers } m \ge 3.$$

Note that for even integers $m \ge 2$ this equality becomes

$$k(D_{2m}) = \frac{m+6}{2}.$$

The equality (9) can be also used to compute the number of conjugacy classes of Dic_3, for which we obtain the following example.

Example 2.7. $k(\mathrm{Dic}_3)=6$.

Finally, we compute the sizes $|O'_{b^ua^v}|$ and $|O_{b^ua^v}|$ of automorphism-conjugacy class and conjugacy class of an arbitrary element $b^ua^v \in G$.

Theorem 2.8. *The following equalities hold*:

(a) $|O'_{b^ua^v}| = \frac{mn\varphi(\frac{(m,[u]_r)}{(m,[u]_r,v)})}{(n,u)(\frac{n}{(n,u)},d)(m,[u]_r)}$;
(b) $|O_{b^ua^v}| = \frac{m}{(m,[u]_r)} o_{\frac{(m,[u]_r)}{(m,[u]_r,v)}}(r)$.

Proof. In both cases, we will apply the orbit-stabilizer theorem. Also, we will use the notation

$$g = \frac{(m,[u]_r)}{(m,[u]_r,v)}.$$

(a) We have

$$|O'_{b^ua^v}| = [\mathrm{Aut}(G) : \mathrm{Stab}_{\mathrm{Aut}(G)}(b^ua^v)].$$

An automorphism $\varphi_{(x_1,x_2,y)}$ is contained in $\mathrm{Stab}_{\mathrm{Aut}(G)}(b^ua^v)$ if and only if $b^ua^v \in \mathrm{Fix}(\varphi_{(x_1,x_2,y)})$, that is if and only if

$$\begin{cases} n \mid u(y-1), \\ m \mid v(x_1-1)+x_2[u]_r. \end{cases}$$

Put $y = zd+1$, where $z \in \{0, 1, \dots, \frac{n}{d}-1\}$. Then the first condition is equivalent with $\frac{n}{(n,u)}|zd$, i.e.

$$\frac{n}{h}\Big|\, z,$$

where $h = (n,u)(\frac{n}{(n,u)}, d)$, and so y can be chosen in $\frac{n}{d} : \frac{n}{h} = \frac{h}{d}$ ways. We also have

$$\begin{aligned}
&|\{(x_1, x_2) \in \overline{0, m-1}^2 \mid (x_1, m) = 1,\ m \mid v(x_1 - 1) + x_2[u]_r\}| \\
&\quad = \sum_{\substack{0 \le x_1 < m \\ (x_1, m)=1}} |\{x_2 \in \overline{0, m-1} \mid m \mid v(x_1 - 1) + x_2[u]_r\}| \\
&\quad = \sum_{\substack{0 \le x_1 < m \\ (x_1, m)=1, g \,|\, x_1 - 1}} (m, [u]_r) \\
&\quad = (m, [u]_r)\frac{\varphi(m)}{\varphi(g)}.
\end{aligned}$$

Thus

$$\begin{aligned}
|O'_{b^u a^v}| &= \frac{|\mathrm{Aut}(G)|}{|\mathrm{Stab}_{\mathrm{Aut}(G)}(b^u a^v)|} \\
&= m\varphi(m)\frac{n}{d} :\ \frac{h}{d}(m, [u]_r)\frac{\varphi(m)}{\varphi(g)} \\
&= \frac{mn\varphi(g)}{h(m, [u]_r)}.
\end{aligned}$$

(b) We have

$$|O_{b^u a^v}| = [\mathrm{Inn}(G) : \mathrm{Stab}_{\mathrm{Inn}(G)}(b^u a^v)]$$

and

$$\begin{aligned}
|\mathrm{Stab}_{\mathrm{Inn}(G)}(b^u a^v)| &= |\{(\alpha, \beta) \in \overline{0, d-1} \times \overline{0, m-1} \mid b^u a^v \in \mathrm{Fix}(\varphi_{(r^\alpha, \beta(1-r), 1)})\}| \\
&= \sum_{0 \le \alpha < d} |\{\beta = \overline{0, m-1} \mid m \mid v(r^\alpha - 1) + \beta(1-r)[u]_r\}| \\
&= \sum_{\substack{0 \le \alpha < d \\ (m, [u]_r) | v(r^\alpha - 1)}} (m, [u]_r) \\
&= \sum_{\substack{0 \le \alpha < d \\ g | r^\alpha - 1}} (m, [u]_r) \\
&= \sum_{\substack{0 \le \alpha < d \\ o_g(r) | \alpha}} (m, [u]_r) \\
&= \frac{d(m, [u]_r)}{o_g(r)}.
\end{aligned}$$

Thus

$$
\begin{aligned}
|O_{b^u a^v}| &= \frac{|\mathrm{Inn}(G)|}{|\mathrm{Stab}_{\mathrm{Inn}(G)}(b^u a^v)|} \\
&= \frac{dm}{\frac{d(m,[u]_r)}{o_g(r)}} \\
&= \frac{m}{(m,[u]_r)} o_g(r)
\end{aligned}
$$

and the proof of our theorem is complete. □

Since $\mathrm{Stab}_{\mathrm{Inn}(G)}(b^u a^v) \cong C_G(b^u a^v)$, the above proof also gives the order of the centralizer of $b^u a^v$ in G.

Acknowledgments

The author is grateful to the reviewer for remarks which improve the previous version of the paper.

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